



The « Barber » Paradox

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[See table of contents](#)

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The "Barber" Paradox

Possibly the most recent allusion in print to this paradox is that found in the article entitled "Paradox," by W. V. Quine, appearing in *Scientific American* for April, 1962 (pp. 84-96). Professor Quine comes to introduce it as follows :

Catastrophe may lurk . . . in the most innocent-seeming paradox. More than once in history the discovery of paradox has been the occasion for major reconstruction at the foundations of thought. For some decades, indeed, studies of the foundations of mathematics have been confounded and greatly stimulated by confrontation with two paradoxes, one propounded by Bertrand Russell in 1901 and the other by Kurt Gödel in 1931.

As a first step onto this dangerous ground, let us consider another paradox [the article having been opened by the relatively facile paradox of Frederic in *The Pirate of Penzance*, who, because he was born on February 29, turns out, after 21 years, to be "a little boy of five"] : that of the village barber. This is not Russell's great paradox of 1901, to which we shall come, but a lesser one that Russell attributed to an unnamed source in 1918. In a certain village there is a man, so the paradox runs, who is a barber ; this barber shaves all and only those men in the village who do not shave themselves. Query : Does the barber shave himself ? (p.84.)

Meanwhile, in the course of attempting to track down the first appearance of the "barber" paradox, Brother E. R. Kiely, F.S.C.H., of Iona College, New Rochelle, N. Y., in a letter received May 14, 1962, obtained from Bertrand Russell these further details : "I did not first propound the paradox of the barber. To the best of my knowledge it was invented by a German called König ; I am not certain of the name, but it was certainly a German. I am afraid that I do not know where it was first published."

What is the precise interest of this paradox ? Professor Quine indicates above the ability of the successful paradox to dictate a whole new direction in thought. He will later indicate the "barber" paradox in particular as being closely related to Russell's paradox of 1901, addressed to Gottlob Frege in connection with the latter's "foundation of mathematics in the self-consistent laws of logic" (p. 90) upon class theory : "For any condition you can formulate, there is a class whose members are the things meeting the condition" (p. 90). Professor Quine relates that Frege's second volume of his *Grundgesetze der Arithmetik* was on its way to press when he received Russell's paradox, relating to self-membership of classes, and that he is supposed to have written in answer : "Arithmetic totters" (p.90). This paradox of Russell is referred to by Professor Quine as "the most celebrated of all

antinomies" (p.90). Of its transformative effect upon reigning class theory, he writes as follows :

... Russell's [antinomy] strikes at the mathematics of classes. Classes are appealed to in an auxiliary way in most branches of mathematics, and increasingly so as passages of mathematical reasoning are made more explicit. The basic principle of classes that is used, at virtually every turn where classes are involved at all, is precisely the class-existence principle that is discredited by Russell's antinomy (p.91).

Consequently, in dealing with the "barber" paradox and its solution, one is dealing with elements of an antinomy which occupies a central position in mathematical logic. "In Russell's antinomy there is more than a hint of the paradox of the barber. The parallel is, in truth, exact" (p.90).

How is the "barber" paradox resolved in Professor Quine's article? First, what is the inextricable situation created by the barber who shaves all those, and only those, in his village who do not shave themselves? It seems that, try as he will, he cannot shave himself :

Any man in this village is shaved by the barber if and only if he is not shaved by himself. Therefore in particular the barber shaves himself if and only if he does not. We are in trouble if we say the barber shaves himself and we are in trouble if we say he does not (p.84).

The solution of Professor Quine is to eliminate the barber :

Happily it [this argument with its unacceptable conclusion] rests on assumptions. We are asked to swallow a story about a village and a man in it who shaves all and only those men in the village who do not shave themselves. This is the source of our trouble ; grant this and we end up saying, absurdly, that the barber shaves himself if and only if he does not. The proper conclusion to draw is just that there is no such barber. We are confronted with nothing more mysterious than what logicians have been referring to for a couple of thousand years as a *reductio ad absurdum*. We disprove the barber by assuming him and deducing the absurdity that he shaves himself if and only if he does not. The paradox is simply a proof that no village can contain a man who shaves all and only those men in it who do not shave themselves. This sweeping denial at first sounds absurd ; why should there not be such a man in the village? But the argument shows why not, and so we acquiesce in the sweeping denial... (p.84).

How does this solution employ the *reductio ad absurdum*? It would do so on the basis that if an initial assumption — in this case, that of a barber in a certain village who shaves all those, and only those, who do not shave themselves — leads to an impossible conclusion, e.g., that a mythical barber must shave himself if he does not, and must not if he does, then that assumption must be rejected as impossible, since, while false assumptions may lead accidentally to true conclusions,

a false and impossible conclusion cannot derive from anything other than a false and impossible assumption. The latter must therefore be rejected and its denial put in its place : Such a barber cannot exist.

Would this be the solution of the "barber" paradox if solved by St. Thomas? It so happens that this query may be answered, since St. Thomas propounds and solves an exactly parallel paradox, that of whether the teacher can teach himself. Where does the similarity lie? Let us first set up the "teacher" paradox in terms similar to those of the "barber" paradox. Thus one could imagine a teacher in a certain village who teaches mathematical logic to all those, and only those, who do not teach themselves. One then has, in the case where the teacher would first have to learn the new discipline of mathematical logic on his own before teaching it to others — a perfectly plausible supposition — a situation exactly comparable to that of the barber, and which could be stated in the same terms with the appropriate substitutions :

Any man in this village is taught mathematical logic by the teacher if and only if he does not teach it to himself. Therefore in particular the teacher teaches himself if and only if he does not. We are in trouble if we say the teacher teaches himself and we are in trouble if we say he does not.

In effect, if the teacher teaches himself, then he should not be taught by the teacher, i. e., himself ; if he does not teach himself, then he needs to be taught by himself.

How does St. Thomas solve the "teacher" paradox, propounded by him in *Quaestiones Disputatae de Veritate*, q.11, a.2 : "Utrum aliquis possit dici magister sui ipsius" ? Just as, in the case of the "barber" paradox, the difficulty lies in the possibility of the barber's being able to shave himself, so in the case of the "teacher" paradox the difficulty lies in the teacher's being able to teach himself. In order for the barber to shave himself in peace and quiet, he would have somehow to cease, for the occasion, to be a barber shaving someone. Even more poignantly, in order to teach himself mathematical logic, the teacher would have, somehow, to cease to be a teacher for the occasion. This is brought out by the fact that, in the same matter, the teacher, as such, cannot be a learner — as he would be if he taught himself. For it is requisite in the teacher that he already have the knowledge which he is to impart, whereas it is precisely the fact of not having it that constitutes the learner a learner. St. Thomas quotes Aristotle's statement in *Physics VIII* (257 a 10) : "... Teaching necessarily implies possessing knowledge, and learning not possessing it." Thus, in the case of the teacher teaching himself, one is in the presence of contradiction, that of simultaneously knowing and not knowing the same thing, perhaps even more vividly than in the case of the barber who cannot shave himself without being simultaneously and contradict-

orily shaved by the barber. The *reductio ad absurdum*, the reduction to the impossible, is as equally in evidence in the first case as in the second. Must the teacher undergo the same fate as the barber, and have to be declared non-existent?

As one knows, this is not St. Thomas' answer. Rather it is that when someone acquires knowledge through his own efforts, even though he be a teacher, he does not do so as a teacher — he does not "teach" himself :

... Without doubt someone can, through the light of reason innate in him, and without the propounding or help of external teaching, arrive at the knowledge of many things he did not know, as is evident in everyone who acquires scientific knowledge through his own discovery. And in this way someone is in a certain sense the cause of his own knowing, but yet he cannot be said to be his own teacher, or to teach himself.¹

Hence the teacher who acquires knowledge through himself need not disappear as an impossible contradiction involving, as being both teacher and learner, the simultaneous possession and ignorance of the knowledge to be acquired. When he teaches others, he is a teacher formally and *per se* ; when he acquires knowledge by himself, although still the same man, he is under that aspect no longer a teacher as such, but a teacher only materially and *per accidens* — as in the case of a violinist who builds a house : as violinist, he plays the violin ; it is not as such that he builds a house.

... Of the causes, a *per se* cause is one type, a *per accidens* cause another. That is said to be a cause *per se* which is the cause of some thing as such — and in this way a builder is the cause of a house, and wood the matter of a bench. That is called a cause *per accidens* which coincides with the cause *per se*, as when we say that a grammarian is building — for this is not insofar as he is a grammarian, but insofar as this happens to be true of a builder ...²

Hence the "teacher" paradox is not solved by eliminating the teacher as an impossibility, but rather by distinguishing between times when the teacher is a teacher and when the teacher is not a teacher, i.e., between the teacher formally and materially, the teacher *per se* and the teacher *per accidens*, the latter being the status of the teacher when he is learning something through his own efforts. Having saved the teacher, or at least half of the teacher, one would like to essay to save the barber in the same way. For just as it does not seem implausible to have a village where a teacher would have to teach mathematical logic to everyone else, while having to learn it himself unaided, so too it seems plausible enough to have a village in which a

1. *Art. cit.*, resp.

2. *De Principiis Naturae*, n.13.

barber shaves all those who are shaveable, and in turn shaves himself. The whole procedure seems quite normal. In the former case he is shaving those who do not shave themselves ; in the latter case he is shaving one who does. But does not this latter case violate the agreed-upon condition that the barber shave only those who do *not* shave themselves ? It does if he is still a barber when he shaves himself. But need he be ? Just as the definition of the teacher as one who already possesses the knowledge being imparted prevents him from teaching himself, so too the definition of the barber as a man who shaves those who do not shave themselves prevents him from barbering himself. When he does that, he is no longer a barber formally and *per se*, but only a barber *per accidens* : he is just a man shaving himself who happens to be a barber.

Just as the situation of the "teacher" may be described in the words Professor Quine uses to describe the predicament of the "barber," so too the words St. Thomas uses to describe how the teacher does not teach himself, may be used to extricate the barber :

Now when someone acquires scientific knowledge [or a shave] through an intrinsic principle [as when the teacher acquires a knowledge of mathematical logic through his own efforts or the barber shaves himself], that which is the agent cause of the knowledge [or the shave] does not have the knowledge to be acquired except in part, namely, as to the seminal reasons of the science, which are the common principles [just as the barber does not have, when shaving himself, a proper client seated before him, but only the basic principles of his art, which he normally uses on such clients]. Therefore in a case of causality of this sort one cannot, properly speaking, apply the name of "teacher" or "master" [or "barber"].¹

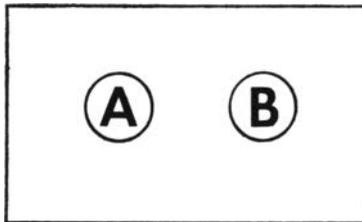
In resolving the "teacher" and the "barber" paradoxes by distinguishing the activities of these two which are *per se* from those which are *per accidens*, thereby allowing for the teacher to be a teacher and not to be a teacher, and the barber to be a barber and not to be a barber, but not under the same aspect, has one committed oneself henceforth never to speak of a teacher, or of anyone, as "teaching himself," or of a barber, or of anyone, as "barbering himself" ? Plainly not. As St. Thomas notes in the last line of his response, we cannot do so "properly speaking" — which implies that we may well do so improperly speaking. In other words we will undoubtedly speak of someone "teaching himself," or even of someone "barbering himself," but in contrast to the formal meaning of these terms, we shall be using them equivocally. Thus an alertness to equivocation, to the use of the same word with different meanings, a phenomenon of common usage due to the paucity of terms as well as to some obvious or remote similarity between the things denoted, would provide one with a first clue in the "barber" and "teacher" paradoxes : things or operations

1. *Art. cit.*, resp., fin. (Matter in brackets added.)

bearing the same name do not need to be essentially the same — allowing for them to be sometimes in apparent contradiction without this actually being the case. Thus when the teacher “teaches himself,” and seems therefore to be in the contradictory status of simultaneously teaching and learning the same thing, of knowing and not knowing the same, this contradiction is resolved by pointing out that “teaching,” as used of teaching others and of teaching oneself, is used equivocally. Thus one may both be and not be the same thing at the same time — because one is not so under the same respect but only equivocally. When the barber shaves himself he is both a barber and not a barber — a barber *per accidens*, and a non-barber *per se* — the same word, “barber,” being used, not univocally, but equivocally, in the two cases. Such a barber can peacefully shave all those, and only those, who do not shave themselves — and then turn and serenely shave himself, because then, in the original univocal sense, he is no longer a barber.

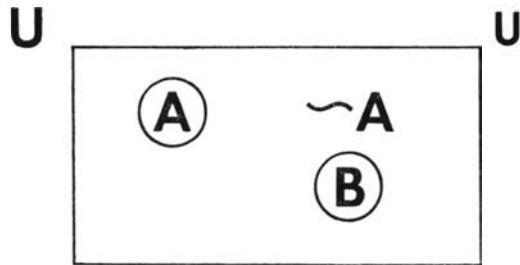
One interesting consequence of the solution of the “barber” paradox in terms of the “barber” *per se* and *per accidens*, formally and materially, actually and potentially, is that it would seem to point to a deficiency in Venn diagrams, namely, their inability to cope with such distinctions.

FIGURE 1



U = All who *can* be clean-shaven
 A = Those shaved by barber
 B = Those shaving selves
 Result : Bearded barber.

FIGURE 2



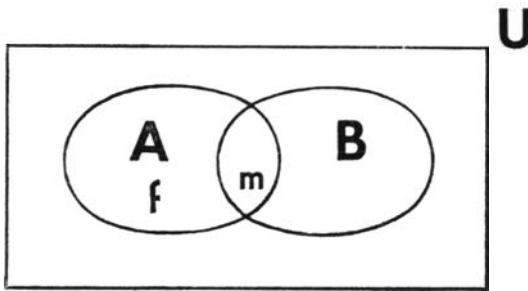
U = All those shaveable in village *are* clean-shaven
 A = Those shaved by barber
 (= $\sim B$)
 $\sim A$ = Those shaving selves
 (= B)
 Result : No barber at all.

Figure 1 : This is the case of a village-universe comprising all who *can* be clean-shaven, allowing therefore that some of these are and some are not. Consequently, if those who *are* clean-shaven are either shaved by the barber (A) or shave themselves (B), it is possible to avoid the predicament of a barber who, it seems, if placed in B, is thereby in A ; or, if placed in A, is thereby in B. One simply places the barber in neither category, but rather in that of the non-shaven,

merely potential, shavers. This would be the complement of the union of A and B. Likewise the barber, since he is in neither A nor B, and these, as exclusive of each other, do not intersect, is in the null set with respect to the intersection of A and B.

Figure 2 : This is the paradox-producing case of the village-universe where all the shaveable members are clean-shaven, and the barber is presumed to be among them. Therefore such a member of the village must be either shaved by the barber (A), or shaved by himself ($\sim A$, or B). In this case the apparent contradiction that immediately arises whether the barber is envisaged in either A or $\sim A$ apparently makes such a barber an impossibility. It would seem that all barbering in a clean-shaven village would have to be done by a visiting barber from another community.

FIGURE 3



U = Same as Figure 2 : all clean-shaven

A = Those shaved by barber

f = formally

m = materially (includes case of barber shaving self : *per accidens* he is being shaved by a barber, but *per se* and formally he is shaving himself)

B = Those shaving selves formally (included self-shaver who happens to be a barber)

Result : Barber shaves self.

Figure 3 : The setting here is the same as that of Figure 2. In this clean-shaven village one has a barber who shaves all those who wish his ministrations, and then proceeds to shave himself as do the other self-shavers. Certainly there seems to be nothing impossible or implausible about this. But this places the barber in an apparent intersection of A and B, which are mutually exclusive ! (He is now a member of a null set with respect to A and B, and becomes a "nothing," in keeping with the conclusion of Figure 2 !) This presence in contradiction could be successfully eliminated if one might explain the barber's membership in an apparently impossible intersection of A an

B as follows : sometimes, when he seems to be in A, he is really in B ; hence his apparent simultaneous presence in both A and B. In effect, when he shaves himself, he only seems to be one of those shaved by the barber ; in reality he is at such a time a non-barber shaving himself. In other words, when shaving himself, he is a non-barber *per se* and formally, and a barber only *per accidens* and materially. Thus, when he shaves himself, it is not really a contradiction of someone being simultaneously shaved by a barber and shaving himself, but rather it is a case of someone shaving himself *per se* and being shaved by a barber only *per accidens*, in that the person shaving himself happens to be a barber. One does not have a simultaneous case, therefore, of A and $\sim A$, of someone being shaved by a barber and not being shaved by a barber in the same respect. Rather it is a case of the simultaneity of someone being shaved by a barber in potency (i.e., someone who could be shaving someone as a barber but is not now doing so), while being shaved by a non-barber in act — in the same sense that water may be simultaneously hot and cold in the sense that the same water which is actually cold simultaneously carries within itself the potency to become hot. The conflict between A and $\sim A$ may therefore be eliminated by placing a different subscript under the two senses of A : “ f ” for “ formally,” “ m ” for “ materially.” It is this very real distinction between two states of a same thing which solves the “ barber ” paradox in a mode which concords with actual experience — but for which the Venn diagram seems to have no allowance.

A TABLE ON EQUIVOCATION

<i>Univocal</i>	<i>Equivocal</i>
One sense only	Simultaneous <i>plurality</i> of senses
=univocal use of word	=equivocal use of word
: the “ proper ” sense of common usage	: “ extended ” sense
: the agent, <i>formally</i> taken, acts <i>per se</i>	: the agent, <i>materially</i> taken, acts <i>per accidens</i>
e.g., “ The violinist (taken formally) plays the violin (<i>per se</i>). ”	e.g., “ The violinist (taken materially) builds (<i>per accidens</i>). ”
— While playing the violin, the violinist, as such, is <i>actual</i> .	— While building, the violinist, as such, is <i>potential</i> .
<i>Formally</i> and <i>per se</i>	<i>Materially</i> and <i>per accidens</i>
TEACHER : teaches <i>others</i> ; he has science, they do not.	“ teaches ” self
BARBER : barbers <i>others</i> , professionally.	“ barbers ” self.

Since it seems that the “ barber ” paradox may have been introduced in 1918 by way of underlining the “ nonself-membership ”

paradox of 1901, "the most celebrated of antinomies," and since "class" theory at present occupies a central position, with Russell's antinomy considered as playing a substantive role in the outlook of that theory, it is appropriate to investigate that antinomy with Professor Quine. What is the logical, if not chronological, continuation of this antinomy with the "barber" paradox? It would seem to lie in the role of paradox in general in forcing us to adjust to new conceptual outlooks. In effect, every paradox comes up with a relatively startling conclusion — otherwise it would not be a paradox, something "outside" common opinion. If the paradox, on closer scrutiny, is seen to have a flaw, the initial surprise disappears, and common opinion is reinstated. Should, on the other hand, however, the conclusion of the paradox prove unassailable, then a revision of previous thinking is in order.

Professor Quine considers both the "barber" paradox and Russell's antinomy to belong to this latter category. In the first case, the paradoxical conclusion to which one appears to be forced by a *reductio ad absurdum*, i.e., by the impossibility of fitting the barber into either of the two exclusive categories, is that there is simply no such barber. But since he was a mythical barber, claiming little credence anyway, this is no great blow and does not provoke any extensive rethinking. "We had never positively believed in such a barber" (p.91). The case of the antinomy on non self-membership of sets, however, is different. It shakes the very foundations of the class theory that is propounded as fundamental in mathematical logic, the latter being in turn extended to all things conceivable (with the universality proper to logic). "Russell's paradox is a genuine antinomy because of the fundamental nature of the principle of class existence it compels us to give up" (*ibid.*).

Where in the previous case the inability of the barber either to shave himself, or to be shaved by a barber, caused his elimination, in the case of Russell's antinomy the impossibility of a class of all classes which are not members of themselves, either to be a member of itself, or not to be a member of itself, decrees the elimination of any such class from class theory — and with it goes the dream of everything being able to be defined in terms of some class, of there being a class for everything (and even for nothing). Apparently there is no class of all classes which are not members of themselves!

The situation leading to Russell's antinomy is described by Professor Quine as follows :

Some classes are members of themselves ; some are not. For example, the class of all classes that have more than five members clearly has more than five classes or members ; therefore the class is a member of itself. On the other hand, the class of all men is not a member of itself, not being a man. What of the class of all classes that are not members of themselves? Since its members are the nonself-members, it qualifies as a member of

itself if and only if it is not. It is and it is not : antinomy's now familiar face.

... I said earlier that an antinomy establishes that some tacit and trusted pattern of reasoning must be made explicit and be henceforward avoided or revised. In the case of Russell's antinomy, the tacit and trusted pattern of reasoning that is found wanting is this : for any condition you can formulate, there is a class whose members are the things meeting the condition.

This principle is not easily given up. The almost invariable way of specifying a class is by stating a necessary and sufficient condition for belonging to it. When we have stated such a condition, we feel that we have "given" the class and can scarcely make sense of there not being such a class. The class may be empty, yes ; but how could there not be such a class at all? What substance can be asked for it that the membership condition does not provide? Yet such exhortations avail us nothing in the face of the antinomy, which simply proves the principle untenable. It is a simple point of logic, once we look at it, that there is no class, empty or otherwise, that has as members precisely the classes that are not members of themselves. It would have to have itself as a member if and only if it did not (p.90).

Hence Frege wrote an appendix to his volume which opens with the words :

"A scientist can hardly encounter anything more undesirable than to have the foundation collapse just as the work is finished. I was put in this position by a letter from Bertrand Russell . . ." (*ibid.*).

The "barber" paradox was solved in terms of the "teacher" paradox by making it possible, through the invoking of the distinction between *per se* and *per accidens*, between formal and material, between actual and potential, for the barber to satisfy the claims upon him by belonging simultaneously, though not in the same respect, to both the conflicting categories. (Taken in the same respect, the categories are of course contradictory and exclusive, and their impossible intersection would be the null set, if the null set may embrace impossible members. Professor Quine, however, in denying the apparently impossible "set of all sets which are not members of themselves" as a null set, apparently restricts the null set to an absence of possible members.) Is it possible to apply the same technique of a dual membership which is simultaneous, but not in the same respect, to the "nonself-membership" paradox?

An initial assumption in the paradox is that of the division of sets into those which are members of themselves and those which are not. Contingent upon this division one then has the paradox of the set of all sets not members of themselves : if that set is not a member of itself, then it should be ; if it is a member of itself, then it should not be, since the sets in question are supposed to be non-members of themselves ; consequently, it seems that no such set can exist.

That is the paradox or antinomy : The hitherto accepted idea of the universal applicability of the set or class concept as a basis of mathematical logic must be relinquished.

It is here proposed to solve this paradox by analysing the idea of "being a member of itself." What does this mean, and is it an illusion? The idea of a set's being a member of itself is clearly derived from the concept of a material collection. Thus, since the set of all sets with more than 5 members must have itself more than 5 members, it is similar to one of its own members and ranks alongside them. The set of all men, however, since it is not an individual man, but a collection of men, cannot rank alongside its members which are individual men (although it can, of course, be an "improper subset" of itself). Because of this, since sets are collections, sets which are members of themselves will necessarily presuppose, for this to be possible, members which are not individuals (e.g., men), but collections (e.g., groups of more than 5). That is, they must be sets of sets. Likewise they cannot, because of the presumed infinity of similar member-sets, be defined in terms of a definite number (e.g., the set of all groups of 5 could *not* be a member of itself).

But is this concept of the set as a collection of individuals, in the same order as its members, not perhaps an illusion? In the case of the set of all sets with more than 5 members, is the set so conceived, like its members, a set of individuals? Or does it not rather abstract from individual characteristics? One can see that the latter must be true, since, in order for the definition of the set to be applicable to every member, the set itself cannot have individual material characteristics in its definition. Thus, supposing it to be a collection of actual material things such as apples, oranges, nuts, bolts, stars, planets, it could apply to some individual member, e.g., six crayfish, only insofar as they agree in being "more than 5" and nothing more. The set, insofar as it might be specifically equated, for example, with planets, could not include as a member a set of six crayfish.

Even if it were restricted to being a set of all sets of more than 5 crayfish, would it be a member of itself? That it would not, as being non-identifiable with any specific member individuals, may be seen by the fact that the set, once defined, would continue to exist even without any existing members of its sets, i.e., should it embrace only the null set. If there were no longer any crayfish at all, the set of all sets of more than 5 crayfish would continue to exist with one member, the null set. This situation is covered by St. Thomas when he says : "If all lions were dead, I could know what 'lion' was [i.e., could conceive of a set of all lions, or of sets of more than 5 lions, even though no lions should happen to be in existence and there would be only a null set of lions]" (*De Veritate*, q.18, a.4, ad 10). The "lions" in the "set of all lions" are not, therefore, necessarily existing lions.

Rather, this set corresponds to nothing other than the *definition* of "lion," abstracting from any individual lion, since it abstracts from the very existence of all lions.

This would be true even if one were to take the "set of all existing things." Such a set would not be defined in terms of individuals, since even if the totality of material existence is represented at a given moment in terms of individuals, nevertheless the mind does not think of it in terms of being restricted to *those* individuals, but as applicable to *any*, even possible, individuals, which might at any time, in the succession of material generation and corruption, constitute the totality of material existence. This is brought out by St. Thomas when he draws attention to the fact that even should there be only one individual in a given species, nevertheless the mind sees it as a nature not restricted to, nor inseparably identified with, that individual, but as able to be found in many :

. . . Every form existing in a singular supposit, through which it is individualized, is common to many, either in actuality, or at least as to notion. Thus human nature is common to many both in fact and as to notion. But the nature of the sun is not common to many in fact, but according to notion only. For the nature of the sun may be understood as existing in several supposites. This is because the intellect understands the nature of every species through abstraction from the singular. Whence, whether something exists in a singular supposit or several is outside the understanding of the nature of a species.¹

The set or class, therefore, in reference to its members, may be rightly identified with a definition, whether of something one *per se* (as "man") or of something one *per accidens* (as a unity of "more 5 members"). (To the extent that a set would be explicitly identified with a collection of existing individuals, e.g., some certain six oranges, then, when those individuals ceased to exist, so would the set. Such sets are not, however, the object of scientific consideration, as may be seen in the concept of the "null set" as a proper subset of all sets, which indicates that the generic "class" or "set" concept abstracts from individual existence.)

In the light of this it is permissible to further analyse in what sense a set may be said to be "a member of itself." The conclusion will be that a set, as a definition, is not a member of itself, but is *in* every member. Thus, in the set of all lions, equivalent to the definition of "lion," this "lion" is found in *every* individual lion. "The name 'lion' is properly communicated to all those in which there is found the nature signified by this name" (St. Thomas, *loc. cit. supra*).

Hence one sees that while no set, as a definition abstracting from the individual, and lacking, therefore, individual characteristics, can be considered, as such, as one of its own members, each of which

1. *Ia*, q.13, a.9, c.

has individual characteristics which set it apart from other members, nevertheless the set is *in* every member — as a specific nature is communicated to a plurality of individuals. Likewise one sees that this is not peculiar to certain sets (e.g., sets of "sets with more than 5 members"), but is common to all sets (e.g., that of "man").

Hence the initial paradox or antinomy, presupposing a dichotomy of sets into sets members of themselves and sets non-members of themselves, would seem to be forestalled by the elimination of the possibility of *any* set being a member of itself. One would recognize, however, that the set is *in* every member.

This conclusion concerning the possibility of any set's being a member of itself may be put to the test by considering the case of the "set of all sets," and that of the "set of all conceivable ideas." It would seem that the "set of all sets" would rightly contain itself as a member, and one would wonder how it could fail to fulfil the definition of "set" applied to its member sets. Yet the universal applicability of the concept of the "set of all sets" to its member sets prevents it itself from being a member set. In effect it must be a concept applicable to every set, and having no additional specific characteristics that would make it true of one set but not of another; the member sets, however, in order to be a plurality, in contrast to the universal set, must have, in addition to the common "set" characteristic, something that sets them off from other sets by making them a specific set. Each member set must have a specific "plus" factor — which is specifically denied to the universal set, and thereby excludes it as a member set. Likewise, although it might seem that the "set of all conceivable ideas," as itself a conceivable idea, must be a member of itself, here too the possibility is forestalled by the fact that such a universal "conceivable idea" must be necessarily devoid of those very specific characteristics which are needed to set one member idea off from another, and which would be needed to constitute it a member alongside them. It is explicitly forbidden to have the very thing that would make it a member.

The concept of the set as being *in* every member, while not a member of itself, provides a single universal basis for set theory in that it provides a single set concept applicable to all things singly or in plurality, namely, that of the set as nothing other than a definition, communicable of its very nature to a plurality of individuals. (These individuals, when existing, may be said to represent the "set in the concrete" — but since such sets have no permanence beyond that of their transient component individuals, they are not the sets of mathematical logic: "Science is not of the individual").

Nor need one despair of finding paradox, since one may say that, while the set is *in* every member, so too every member is *in* the set. One might thus suggest the paradox that the set cannot be *in* the member, since the member must be *in* the set, and conversely. This, of

course, is solved by adverting to the similar manner in which, simultaneously, the whole may be in the part, and the part in the whole. In set theory, the whole is in the part insofar as the definition which identifies the set is found fulfilled in every individual ; the part is in the whole insofar as each individual member is contained under the definition of the set which, by its very nature, as seen above, is applicable to, and embraces, a plurality of members. This mutualness is comparable to the manner in which the genus is in its species (as " animal " is in " man " and " tiger ") and the species is in its individuals (as " man " is in Tom, Dick and Harry), while conversely the species is part of a larger genus and the individual is a material part of a species :

... [Aristotle] sets down four ways in which something is called a " part."

... In the second way, those things are called " parts " into which something is divided without quantity. And in this way species are said to be a part of the genus [as would be the specific member sets of the generic set of all sets with more than 5 members]. For it [the genus] is divided into species, not as quantity into quantitative parts. For the whole quantity is not in each of its parts — but the genus is in each of the species.

... In the fourth way, those things are called " parts " which are parts of the definition, as " animal " and " biped " are parts of " man."

From this it is evident that genus in the fourth sense is part of the species : but in another way, namely, the second way, the species is part of the genus.¹

(The way the species is in the individual as the universal part thereof, and the individual is in the species as a quasi-material part thereof seems sufficiently clear of itself. When the universal set is a set of sets and the members are sets, one has the genus-species relationship ; when the universal set has individual members, one has the species-individual relationship. In both cases the member represents, vis-à-vis its universal set, a plurality-permitting determination which is explicitly denied the universal set. The parallel between genus-species and species-individual is seen in the fact that the species, vis-à-vis the genus, is sometimes referred to as an " individual.") Because purely immaterial beings can neither be considered properly as species of a genus, nor individuals of a species, they thereby cannot be properly made members of any defined set. Conversely, this obviously restricts set theory, properly speaking, to the scope of the material.

Might one, in connection with sets and their members, invoke here the distinction of *per se* and *per accidens*, as was done in the case of the " barber " ? He was, when shaving himself, *per se* a self-shaver and only *per accidens* a barber. What of the set in the member and the member in the set ? It seems that one might say that the set

1. ST. THOMAS, *In V Metaph*, lect.21, nn.1093-1097.

tends to be *per se* in the member, in that the genus becomes more actualized in the species, and the species in the individual, whereas the member tends to be, by contrast, *per accidens* in the set, insofar as being there in an abstract, universalized state. "... Species ... are more known by nature, as more perfectly existing and being the object of distinct knowledge. Genera, however, are prior known to us, as being the object of knowledge in potency and confused."¹ Significantly, in the passage of Aristotle which St. Thomas is here expounding, the species are called "individuals."

It would seem that the whole question of "class" or "set" theory might be brought into a more satisfying perspective if examined in the light of the distinctions employed by Aristotle and St. Thomas. Possibly the most fundamental of these as applied to set theory is that between the material singular and the conceptual universal. The paradox of the set of all nonself-member sets would seem to take its rise basically from an initial refusal to distinguish between the set in the concrete, potentially constituted out of a certain number of material individuals, and the set as a universal concept, expressed in a definition, and of which each member is a fulfilment.

(This seems to be revealed in the concept of "mathematical induction," whose elements, such as "number," "0," "successor," are in turn defined by Bertrand Russell in terms of "sets" or "classes." This "induction" is seen as replacing, as a kind of direct mental intuition or assumption, the induction based on going from singular cases to a universal. Considering such an induction as a prelude to sets or classes, one would be tempted to say that mathematical logic makes normal induction without realizing or acknowledging it, and because of this does not realize that the sets and classes with which it deals are actually universals and not collections of singulars. The fact that for mathematical universals a minimum of induction is necessary leads to this deception.)

As a consequence, the set, really a universal, is simultaneously treated as though it were a collection of individuals, and considered in certain cases a member of itself along with the member-collections — thereby leading to paradox. A concept of set as a defined universal, however, provides an all-embracing basis for set operations.

This may be seen in its adequate accounting for concepts such as that of the "improper subset" as containing all the members of the set (and therefore "improper" as being thus indistinguishable from the set itself considered as a collection of individuals), and of the "null set." In effect, should one take any material species or part thereof, e.g., the "set of three eggs in a nest," due to the fact that in the processes of nature things do not succeed with necessity, the subsets of eggs which will hatch may be anything from all the eggs, an

1. ST. THOMAS, *In I Phys.*, lect.1, n.7.

“improper subset,” or none, the “null set.” It is precisely because of this indeterminacy connected with matter that any realistic representation of things and events requires the distinction between necessary and contingent as above ; between potency and act, whereby a thing, while being one thing at the moment, e.g., hot, is nevertheless at the same time potentially some other thing, e.g., cold ; between “formally” and “materially,” whereby the same subject which is a doctor medicating formally, may at the same time, by virtue of an identity of subject, be a musician medicating materially ; when such a doctor medicates, it is the doctor medicating *per se*, and the musician which he happens to be, medicating *per accidens*.

Plainly all these “ambivalent” traits stem in some way from the indeterminacy peculiar to material beings, expressed in the fact that no one form exhausts the continuing potentiality of matter to other forms — when a thing is actually under one form, it is potentially able to be under another. The fact that the material individual’s operation at any given moment is relatively limited means that while it is formally one thing, it is materially or potentially others ; while it is one thing *per se*, it is many *per accidens*.

Any representation of the material individuals composing sensible reality which does not allow for their intrinsic mutability, as any purely mathematical representation which freezes things into permanent categories does not, will necessarily prove inadequate. If the ambivalent potentiality whereby something may belong simultaneously to two opposing categories is eliminated, one runs into paradoxes of barbers who can’t be either of two things because to do so they would have to be both (as they actually are in reality). Does this nullify the suitability of mathematical representations? No ; but it does stress the need for the introduction of distinctions. Otherwise, the price of simplicity is the sacrifice of veridical representation.

Pierre H. CONWAY.

