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Article abstract

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# Polynomial Interior Point Algorithms for General Linear Complementarity Problems

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**Abstract**

*Linear Complementarity Problems (LCPs) belong to the class of $\mathbb{NP}$-complete problems. Therefore we can not expect a polynomial time solution method for LCPs without requiring some special property of the coefficient matrix. Following our recently published ideas we generalize affine scaling and predictor-corrector interior point algorithms to solve LCPs with general matrices in EP-sense, namely, our generalized interior point algorithms either solve the problems with rational coefficient matrix in polynomial time or give a polynomial size certificate that our matrix does not belong to the set of $\mathcal{P}_*(\tilde{\kappa})$ matrices, with arbitrary large, but apriori fixed, rational, positive $\tilde{\kappa}$.*

*Key words:* linear complementarity problem, sufficient matrix, $\mathcal{P}_*$-matrix, interior point method, affine scaling method, predictor-corrector algorithm.

## 1. Introduction

Consider the *Linear Complementarity Problem* (LCP): find vectors $\mathbf{x},\ \mathbf{s} \in \mathbb{R}^n$ that satisfy

$$-M\mathbf{x} + \mathbf{s} = \mathbf{q}, \quad \mathbf{x}\mathbf{s} = \mathbf{0}, \quad \mathbf{x},\ \mathbf{s} \geq \mathbf{0}, \qquad (1)$$

where $M \in \mathbb{R}^{n\times n}$ and $\mathbf{q} \in \mathbb{R}^n$, and the notation $\mathbf{xs}$ is used for the coordinatewise (Hadamard) product of the vectors $\mathbf{x}$ and $\mathbf{s}$.

LCPs belong to the class of $\mathbb{NP}$ hard problems, since the feasibility problem of linear equations with binary variables can be described as an LCP [14]. Therefore we can not expect an efficient (polynomial time) solution method for LCPs without requiring some special property of the matrix $M$.

In [9] we modified long-step path-following *Interior Point Methods* (IPMs) for LCPs with a general coefficient matrix $M$. The modified algorithm either solves the LCP, or gives a certificate that the matrix of the problem is not $\mathcal{P}_*(\tilde{\kappa})$ (with apriori given but arbitrary large $\tilde{\kappa}$), or gives a certificate that the LCP has no solution. Algorithms that do not require any special property of the Matrix $M$ are needed becasue it cannot be verified in polynomial time if matrix $M$ belongs to the class of matrices that allow polynomial time solvability of the LCP. Indeed, Tseng [21], proved that the problem deciding whether a square matrix with rational entries is a column sufficient matrix is co-$\mathbb{NP}$-complete, suggesting that it can not be decided in polynomial time whether there is a finite nonnegative $\kappa$ with which matrix $M$ is $\mathcal{P}_*(\kappa)$. In this paper we show, that the idea behind the modification of long-step path-following algorithm is general, i.e., it can be adapted to other IPMs as well. Thus here we present the modifications of two well known IPMs: the affine scaling and predictor corrector algorithms.

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In [9] first we discussed the generalization of the embedding technique of Kojima et al. [14]. The embedding technique is one of the options to ensure the availability of an initial interior point. This technique is independent of the particular IPM, therefore we do not repeat it in this paper. Thus we may assume without loss of generality that an initial interior feasible solution is known.

The rest of the paper is organized as follows. The next section deals with the fundamental properties of $\mathcal{P}_*(\kappa)$-matrices and some well-known results are presented. In Section 3 we summarize the results of paper [9]. Section 4 deals with the modification of two well known interior point algorithms: the affine scaling and predictor-corrector algorithms.

For ease of understanding and for self containedness the main results of the papers [11,18] are summarized in the Appendix.

**Notation:**
We use the following notations throughout the paper. Scalars and indices are denoted by lowercase Latin letters, vectors by lowercase boldface Latin letters, matrices by capital Latin letters, and finally sets by capital calligraphic letters. Let $\mathbb{R}^n_\oplus$ ($\mathbb{R}^n_+$) denote the nonnegative (positive) orthant of $\mathbb{R}^n$. Further, $I$ denotes the identity matrix of appropriate dimension, and $X$ is the diagonal matrix whose diagonal elements are the coordinates of the vector $\mathbf{x}$, so $X = \mathrm{diag}(\mathbf{x})$. The vector $\mathbf{x}\,\mathbf{s} = X\mathbf{s}$ is the componentwise product (Hadamard product) of the vectors $\mathbf{x}$ and $\mathbf{s}$, and for $\alpha \in \mathbb{R}$ the vector $\mathbf{x}^\alpha$ denotes the vector whose $i$th component is $x_i^\alpha$. The largest and smallest coordinate of a vector is denoted by $\max(\mathbf{x})$ and $\min(\mathbf{x})$, respectively. We denote the vector of ones by $\mathbf{e}$. Furthermore, for a vector $\mathbf{x}$ and a matrix $M$ we define the sets $\mathcal{I}_+(\mathbf{x}) = \{1 \le i \le n : \ x_i(Mx)_i > 0\}$ and $\mathcal{I}_-(\mathbf{x}) = \{1 \le i \ \le n : \ x_i(Mx)_i < 0\}$, which are used in the definition of $\mathcal{P}_*(\kappa)$ matrices. Finally, $\mathcal{F}^0 := \left\{(\mathbf{x},\, \mathbf{s}) \in \mathbb{R}^{2n}_+ : -M\mathbf{x} + \mathbf{s} = \mathbf{q}\right\}$ denotes the set of strictly feasible solutions of the LCP.

## 2. Matrix classes and the Newton step

The class of $\mathcal{P}_*(\kappa)$-matrices were introduced by Kojima et al. [14], and it can be considered as a generalization of the class of positive semidefinite matrices.

**Definition 1** *Let $\kappa \ge 0$ be a nonnegative number. A matrix $M \in \mathbb{R}^{n\times n}$ is a* $\mathcal{P}_*(\kappa)$-matrix *if for all $\mathbf{x} \in \mathbb{R}^n$*

$$(1+4\kappa) \sum_{i\in\mathcal{I}_+(\mathbf{x})} x_i(M\mathbf{x})_i + \sum_{i\in\mathcal{I}_-(\mathbf{x})} x_i(M\mathbf{x})_i \ge 0. \quad (2)$$

The nonnegative real number $\kappa$ denotes the weight that need to be used at the positive terms so that the weighted 'scalar product' is nonnegative for each vector $\mathbf{x} \in \mathbb{R}^n$. Therefore, naturally $\mathcal{P}_*(0)$ is the class of positive semidefinite matrices (setting aside the symmetry of the matrix $M$).

**Definition 2** *A matrix $M \in \mathbb{R}^{n\times n}$ is called a $\mathcal{P}_*$-*matrix *if it is a $\mathcal{P}_*(\kappa)$-matrix for some $\kappa \ge 0$, i.e.*

$$\mathcal{P}_* = \bigcup_{\kappa\ge 0} \mathcal{P}_*(\kappa).$$

The class of sufficient matrices was introduced by Cottle, Pang and Venkateswaran [3].

**Definition 3** *A matrix $M \in \mathbb{R}^{n\times n}$ is a* column sufficient matrix *if for all $\mathbf{x} \in \mathbb{R}^n$*

$$X(M\mathbf{x}) \le 0 \ \ \textit{implies} \ \ X(M\mathbf{x}) = 0,$$

*and* row sufficient *if $M^T$ is column sufficient. Matrix $M$ is* sufficient *if it is both row and column sufficient.*

Kojima et al. [14] proved that any $\mathcal{P}_*$-matrix is column sufficient and Guu and Cottle [7] proved that it is row sufficient, too. Therefore, each $\mathcal{P}_*$-matrix is sufficient. Väliaho proved the other direction of inclusion [22], thus the class of $\mathcal{P}_*$-matrices coincides with the class of sufficient matrices.

**Definition 4** *A matrix $M \in \mathbb{R}^{n\times n}$ is a* $\mathcal{P}_0$-matrix, *if all of its principal minors are nonnegative.*

For further use we recall some results about $\mathcal{P}_*(\kappa)$- and $\mathcal{P}_0$-matrices. The reader may consult the book of Kojima et al. [14, Lemma 4.1 p. 35] for the proof of the following proposition.

**Proposition 5** *A matrix $M \in \mathbb{R}^{n\times n}$ is a $\mathcal{P}_0$-matrix if and only if*

$$M' = \begin{bmatrix} -M & I \\ S & X \end{bmatrix} \ \textit{is a nonsingular matrix}$$

*for any positive diagonal matrices $X, S \in \mathbb{R}^{n\times n}$.* □

Proposition 5 enables us to check in strongly polynomial time whether matrix $M$ is $\mathcal{P}_0$ or not. The next statement is used to guarantee the existence and uniqueness of Newton directions that are the solution of system (3) for various values of vector $\mathbf{a} \in \mathbb{R}^n$, where $\mathbf{a}$ depends on the particular IPM.

**Corollary 6** *Let $M \in \mathbb{R}^{n\times n}$ be a $\mathcal{P}_0$-matrix, $(\mathbf{x},\mathbf{s}) \in \mathcal{F}^0$. Then, for all $\mathbf{a} \in \mathbb{R}^n$ the system*

$$\begin{aligned} -M\boldsymbol{\Delta}\mathbf{x} + \boldsymbol{\Delta}\mathbf{s} &= \mathbf{0} \\ \mathbf{s}\boldsymbol{\Delta}\mathbf{x} + \mathbf{x}\boldsymbol{\Delta}\mathbf{s} &= \mathbf{a} \end{aligned} \quad (3)$$

*has a unique solution $(\boldsymbol{\Delta}\mathbf{x}, \boldsymbol{\Delta}\mathbf{s})$.* □

The following estimations for the Newton direction are used in the complexity analysis of IPMs. The next lemmas are proved by Potra in [17].

**Lemma 7** *Let $(\mathbf{x},\mathbf{s}) \in \mathcal{F}^0$ and $M$ be an arbitrary $n\times n$ real matrix and $(\Delta\mathbf{x},\Delta\mathbf{s})$ be a solution of system (3). Then*

$$\sum_{i\in\mathcal{I}_+} \Delta x_i \Delta s_i \le \frac{1}{4}\left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2.$$

**Lemma 8** *Let the matrix $M$ be a $\mathcal{P}_*(\kappa)$-matrix, $\mathbf{x},\mathbf{s} \in \mathcal{F}^0$, and $\mathbf{a}\in\mathbb{R}^n$. Let $(\Delta\mathbf{x},\Delta\mathbf{s})$ be the solution of system (3). Then*

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_\infty \le \left(\frac{1}{4}+\kappa\right)\left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2,$$

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_1 \le \left(\frac{1}{2}+\kappa\right)\left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2,$$

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_2 \le \sqrt{\left(\frac{1}{4}+\kappa\right)\left(\frac{1}{2}+\kappa\right)}\left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2.$$

The first statement's proof in the previous lemma is similar to the proof of Lemma 5.1 by Illés, Roos and Terlaky [11]. The second estimation follows from the Lemma 7 by using some properties of $\mathcal{P}_*(\kappa)$-matrices, and the last estimation is a corollary of the first and second statements using some properties of norms.

Let the current point be $(\mathbf{x},\mathbf{s}) \in \mathcal{F}^0$ and $(\Delta\mathbf{x},\Delta\mathbf{s})$ be the corresponding Newton direction. The new point with step length $\theta$ is given by $(\mathbf{x}(\theta),\mathbf{s}(\theta)) = (\mathbf{x}+\theta\Delta\mathbf{x}, \mathbf{s}+\theta\Delta\mathbf{s})$. We use the following notations for scaling

$$\mathbf{v} = \sqrt{\frac{\mathbf{xs}}{\mu}}, \quad \mathbf{g} = \frac{\Delta\mathbf{x}\Delta\mathbf{s}}{\mu}, \tag{4}$$

where in the affine scaling algorithm for the purpose of scaling we have $\mu \equiv 1$, otherwise $\mu > 0$. In affine scaling algorithms we use the $\delta_a$ centrality measure

$$\delta_a(\mathbf{xs}) = \frac{\max\left(\sqrt{\mathbf{xs}}\right)}{\min\left(\sqrt{\mathbf{xs}}\right)}.$$

In the predictor-corrector algorithm the so-called negative infinity neighborhood $\mathcal{D}(\gamma)$, defined by Potra and Liu in [18], is used. The negative infinity neighborhood for $\gamma \in (0,1)$ is defined as

$$\mathcal{D}(\gamma) := \left\{(\mathbf{x},\,\mathbf{s}) \in \mathcal{F}^0 :\ \mathbf{x}\,\mathbf{s} \ge \gamma\,\frac{\mathbf{x}^T\mathbf{s}}{n}\right\},$$

where $\mathcal{F}^0 := \left\{(\mathbf{x},\,\mathbf{s}) \in \mathbb{R}^{2n}_+ : -M\mathbf{x}+\mathbf{s} = \mathbf{q}\right\}$ denotes the set of strictly feasible solutions of the LCP. The $\mathcal{D}(\gamma)$ neighborhood is considered to be a "wide neighborhood".

## 3. Previous results

In this section we restate those results of paper [9], that we use in the rest of this paper.

The inequality in the definition of $\mathcal{P}_*(\kappa)$-matrices gives the following lower bound on $\kappa$ for any vector $\mathbf{x}\in\mathbb{R}^n$:

$$\kappa \ge \kappa(\mathbf{x}) = -\frac{1}{4}\frac{\mathbf{x}^T M\mathbf{x}}{\sum_{i\in\mathcal{I}_+} x_i(Mx)_i}.$$

The following two lemmas are immediate consequences of the definition of $\mathcal{P}_*(\kappa)$ and $\mathcal{P}_*$-matrices.

**Lemma 9** *Let $M$ be a real $n\times n$ matrix and $\tilde\kappa > 0$ be a given parameter. If there exists a vector $\mathbf{x}\in\mathbb{R}^n$ such that $\kappa(\mathbf{x}) > \tilde\kappa$, then the matrix $M$ is not $\mathcal{P}_*(\tilde\kappa)$ and $\mathbf{x}$ is a certificate for this fact.*

**Lemma 10** *Let $M$ be a real $n\times n$ matrix. If there exists a vector $\mathbf{x}\in\mathbb{R}^n$ such that $\mathcal{I}_+(\mathbf{x}) = \{i\in\mathcal{I} : x_i(Mx)_i > 0\} = \emptyset$, then the matrix $M$ is not $\mathcal{P}_*$ and $\mathbf{x}$ is a certificate for this fact.*

Our modified interior point algorithms are based on the assumption that the problem data is rational, and we are interested either to provide a polynomial size solution to the problem, or to provide a certificate that our matrix $M$ is not a $\mathcal{P}_*(\bar\kappa)$-matrix for some $\bar\kappa > 0$. During all iterations of our algorithm we have an actual estimate for $\kappa$, say $\bar\kappa$. At the beginning of our algorithms, we assume that $\bar\kappa = 0$. In every iteration at different stages of the algorithm we are checking whether our assumption related to $\kappa$ is violated. If we detect, that the matrix $M$ is not $\mathcal{P}_*(\bar\kappa)$ with actual $\bar\kappa$, then the value of $\bar\kappa$ will be increased. Its new value will be the lower bound defined by the actual Newton direction $\Delta\mathbf{x}$. In IPMs the $\mathcal{P}_*(\kappa)$ property need to hold only for the actual Newton direction $\Delta\mathbf{x}$ in various ways, for example this property ensures that with a certain step size the new iterate is in an appropriate neighborhood of the central path and/or the complementarity gap is sufficiently reduced. Consequently, if the desired results do not hold with the current $\bar\kappa$ value, we update $\bar\kappa$ by increasing it to the lower bound determined by the Newton direction $\Delta\mathbf{x}$, i.e.,

$$\bar\kappa = \kappa(\Delta\mathbf{x}) =: \frac{-\Delta\mathbf{x}^T\Delta\mathbf{s}}{4\sum_{i\in\mathcal{I}_+}\Delta x_i\Delta s_i} \quad (\Delta\mathbf{s} = M\Delta\mathbf{x}). \tag{5}$$

The following lemma is our main tool to verify when matrix $M$ fails to satisfy the $\mathcal{P}_*(\kappa)$ property. Furthermore, the concerned vector $\Delta\mathbf{x}$ is a certificate, whose encoding size is polynomial when it is computed as the solution of the Newton system (3) from rational data.

We use this lemma during the analysis. The first statement is simply the negation of the definition. We point out in Lemma 12 that if Lemma 4.3 of [12] does not hold, then the second statement is realized. We show in Lemma 15 and Lemma 16 that if Theorem 3.3 of [18] does not hold then the second, the third or the fourth statement is realized.

**Lemma 11** *Let $M$ be a real $n \times n$ matrix, $\kappa \geq 0$ be a given parameter, and $(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^0$. If any of the following statements holds then the matrix $M$ is not a $\mathcal{P}_*(\kappa)$-matrix.*

*(1) There exists a vector $\mathbf{y} \in \mathbb{R}^n$ such that*

$$(1+4\kappa) \sum_{i \in \mathcal{I}_+(\mathbf{y})} y_i w_i + \sum_{i \in \mathcal{I}_-(\mathbf{y})} y_i w_i < 0,$$

*where $\mathbf{w} = M\mathbf{y}$ and $\mathcal{I}_+(\mathbf{y}) = \{i \in I : y_i w_i > 0\}$, $\mathcal{I}_-(\mathbf{y}) = \{i \in I : y_i w_i < 0\}$.*

*(2) There exists a solution $(\Delta\mathbf{x}, \Delta\mathbf{s})$ of system (3) such that*

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_\infty > \frac{1+4\kappa}{4} \left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}} \right\|^2.$$

*(3) There exists a solution $(\Delta\mathbf{x}, \Delta\mathbf{s})$ of system (3) such that*

$$\max\left( \sum_{i \in \mathcal{I}_+} \Delta x_i \Delta s_i,\ -\sum_{i \in \mathcal{I}_-} \Delta x_i \Delta s_i \right) > \frac{1+4\kappa}{4} \left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}} \right\|^2.$$

*(4) There exists a solution $(\Delta\mathbf{x}, \Delta\mathbf{s})$ of system (3) such that*

$$\Delta x^T \Delta s < -\kappa \left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}} \right\|^2.$$

## 4. Interior point algorithms in EP form

Hereafter we modify the following two popular families of IPMs:

- A family of affine scaling algorithms [11]:
  The Newton direction is the solution of system (3) with $\mu = 1$ and $\mathbf{a} = -\frac{\mathbf{v}^{2r+2}}{\|\mathbf{v}^{2r}\|}$, where $r \geq 0$ is the degree of the algorithm.
  The choice of $\mathbf{a}$ implies $\left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}} \right\|^2 = \frac{\left\|\mathbf{v}^{2r+1}\right\|^2}{\|\mathbf{v}^{2r}\|^2}$.
- Predictor-corrector algorithms [18]:
  The predictor Newton direction is the solution of system (3) with $\mathbf{a} = -\mathbf{xs}$ (the affine scaling direction with $r = 0$).
  The choice of $\mathbf{a}$ implies $\left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}} \right\|^2 = \mathbf{x}^T\mathbf{s}$;
  The corrector (centering) Newton direction is the solution of system (3) with $\mathbf{a} = \mu\mathbf{e} - \mathbf{xs}$, where $\mu = \frac{\mathbf{x}^T\mathbf{s}}{n}$.
  The choice of $\mathbf{a}$ implies

$$\left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}} \right\|^2 = \mu \left\| \sqrt{\frac{\mathbf{xs}}{\mu}} - \sqrt{\frac{\mu}{\mathbf{xs}}} \right\|^2.$$

### 4.1. *Affine scaling IPMs*

First we deal with affine scaling IPMs. We modify the family of algorithms proposed in [11], where the particular algorithms correspond to the degree $r \geq 0$ of the algorithm, where $r = 0$ gives the classical primal-dual affine scaling algorithm, while $r = 1$ gives the primal-dual Dikin affine scaling algorithm [1]. Further, there is a step length parameter $\nu$, that depends on the degree $r$ (defined among the inputs of the algorithm), and $\mu \equiv 1$ in scaling (4).

We check not only the solvability and uniqueness of the Newton system, but also the decrease of the complementarity gap after a step. For the actual value of $\kappa$ we determine $\theta_a^*(\kappa)$, which is a theoretical lower bound for the maximal feasible step length in the specified neighborhood if the matrix $M$ satisfies the $\mathcal{P}_*(\kappa)$ property. Therefore, if after a step the decrease of the complementarity gap is not large enough, it means, that the matrix $M$ is not $\mathcal{P}_*(\kappa)$ with the actual value of $\kappa$, so we update $\kappa$ or exit the algorithm with a proper certificate. If the new value of $\kappa$ can not be defined by (5), then the matrix $M$ is not $\mathcal{P}_*$, so we stop and the Newton direction $\Delta\mathbf{x}$ is a certificate. If the new value of $\kappa$ is larger than $\tilde{\kappa}$, then the matrix is not $\mathcal{P}_*(\tilde{\kappa})$, therefore the algorithm stops as well and $\Delta\mathbf{x}$ is a certificate. In the rest of this subsection we consider the case $r > 0$. The modified algorithm is presented in Algorithm 1.

Illés et al. proved [11], that if the matrix $M$ is a $\mathcal{P}_*(\kappa)$-matrix, then the step length $\theta_a^*(\kappa)$ is feasible, with that step size the new iterate stays within the specified neighborhood and it provides the required decrease of the complementarity gap. The following lemma shows, if the decrease of the complementarity

**Algorithm 1 - Affine scaling algorithm**

**Input:**

an upper bound $\tilde{\kappa} > 0$ on the value of $\kappa$;
an accuracy parameter $\varepsilon > 0$;
a centrality parameter $\tau$;
the degree of scaling $r > 0$;
a strictly feasible initial point $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{F}^0$ such that $\delta_a(\mathbf{x}^0\,\mathbf{s}^0) \le \tau$;

$$\nu := \begin{cases} 2/\sqrt{n}, & \text{if } 0 < r \le 1 \\ 2\tau^{2-2r}/\sqrt{n}, & \text{if } 1 \le r; \end{cases}$$

$$\theta_a^*(\kappa) := \min\left\{ \frac{2}{(1+4\kappa)\tau}\left(\sqrt{1+4\kappa+\frac{1}{\tau^2\,n}} - \frac{1}{\tau\sqrt{n}}\right), \frac{\sqrt{n}}{(r+1)\tau^{2r}}, \frac{4(\tau^{2r}-1)}{(1+4\kappa)(1+\tau^2)\tau^{2r}\sqrt{n}}, \nu \right\}.$$

**begin**

$\mathbf{x} := \mathbf{x}^0$, $\mathbf{s} := \mathbf{s}^0$, $\kappa := 0$;
**while** $\mathbf{x}^T\mathbf{s} \ge \varepsilon$ **do**
calculate the Newton direction $(\Delta\mathbf{x}, \Delta\mathbf{s})$ with $\mathbf{a} = -\mathbf{v}^{2r+2}/\|\mathbf{v}^{2r}\|$;
**if** (the Newton direction does not exist or it is not unique) **then**
**return** the matrix is not $\mathcal{P}_0$; *% see Corollary 6*
$\bar{\theta} = \operatorname{argmin}\left\{\mathbf{x}(\theta)^{\mathrm{T}}\mathbf{s}(\theta) : \ \delta_{\mathrm{a}}\left(\mathbf{x}(\theta), \mathbf{s}(\theta)\right) \le \tau,\ (\mathbf{x}(\theta), \mathbf{s}(\theta)) \ge \mathbf{0}\right\}$;
**if** $\left(\mathbf{x}(\bar{\theta})^T\mathbf{s}(\bar{\theta}) > (1 - 0.25\,\nu\,\theta_a^*(\kappa))\,\mathbf{x}^T\mathbf{s}\right)$ **then**
calculate $\kappa(\Delta\mathbf{x})$; *% see (5)*
**if** ($\kappa(\Delta\mathbf{x})$ is not defined) **then**
**return** the matrix is not $\mathcal{P}_*$; *% see Lemma 10*
**if** ($\kappa(\Delta\mathbf{x}) > \tilde{\kappa}$) **then**
**return** the matrix is not $\mathcal{P}_*(\tilde{\kappa})$; *% see Lemma 9*
$\kappa = \kappa(\Delta\mathbf{x})$;
update $\theta_a^*(\kappa)$; *% it depends on $\kappa$*
$\mathbf{x} = \mathbf{x}(\bar{\theta}),\ \mathbf{s} = \mathbf{s}(\bar{\theta})$;
**end**
**end.**

gap is not sufficient, then the matrix $M$ does not belong to the class of $\mathcal{P}_*(\kappa)$-matrices.

**Lemma 12** *If $\mathbf{x}(\bar{\theta})^T\mathbf{s}(\bar{\theta}) > (1 - 0.25\,\nu\,\theta_a^*(\kappa))\,\mathbf{x}^T\mathbf{s}$, that is, the decrease of the complementarity gap within the $\delta_a \le \tau$ neighborhood is not sufficient, then the matrix $M$ of the LCP is not $\mathcal{P}_*(\kappa)$ with the actual value of $\kappa$. The Newton direction $\Delta\mathbf{x}$ serves as a certificate.*

**Proof:** Based on [12, Lem 4.3] (see Lemma 20 in the Appendix) the complementarity gap at $\theta_a^*(\kappa)$ is smaller than $(1 - 0.25\,\nu\,\theta_a^*(\kappa))\,\mathbf{x}^T\mathbf{s}$, furthermore by Theorem 21, if $M$ is a $\mathcal{P}_*(\kappa)$-matrix, then the point $(\mathbf{x}^*, \mathbf{s}^*) = (\mathbf{x}(\theta_a^*(\kappa)), \mathbf{s}(\theta_a^*(\kappa)))$ is feasible. Therefore, if $\mathbf{x}(\bar{\theta})^T\mathbf{s}(\bar{\theta}) > (1 - 0.25\,\nu\,\theta_a^*(\kappa))\,\mathbf{x}^T\mathbf{s}$, then because the step length $\theta_a^*(\kappa)$ is not considered in definition of $\bar{\theta}$ (see the affine scaling algorithm), so either $(\mathbf{x}^*, \mathbf{s}^*)$ is not feasible, or this point is not in the $\tau$ neighborhood of the central path, namely $\delta_a(\mathbf{x}^*\mathbf{s}^*) > \tau$. We show that both cases imply, that the matrix $M$ is not $\mathcal{P}_*(\kappa)$ with the actual $\kappa$ value.

Let us denote the first three terms in the definition of $\theta_a^*(\kappa)$ by $\theta_1$, $\theta_2$, and $\theta_3$, respectively. We follow the proof of Theorem 6.1 in [11] (see Theorem 21 in the Appendix). We need to reconsider only the expressions depending on $\kappa$. There exist positive constants $\alpha$ and $\beta$ such that $\frac{\beta}{\alpha} = \tau^2$ and $\alpha\mathbf{e} \le \mathbf{v}^2 \le \beta\mathbf{e}$. The function $\varphi(t) = t - \theta\frac{t^{r+1}}{\|\mathbf{v}^{2r}\|}$ remains monotonically increasing on the interval $[0, \beta]$ if $\theta \le \theta_2 \le \frac{\|\mathbf{v}^{2r}\|}{(r+1)\beta^r}$. Additionally, inequalities (17) in [11] hold for $\theta \le \theta_2$, thus for $\theta_a^*(\kappa)$ too:

$$\min(\mathbf{v}^{*2}) \ge \alpha - \theta_a^*(\kappa)\frac{\alpha^{r+1}}{\|\mathbf{v}^{2r}\|} - (\theta_a^*(\kappa))^2\|\mathbf{g}\|_\infty, \quad (6)$$

$$\max(\mathbf{v}^{*2}) \le \beta - \theta_a^*(\kappa)\frac{\beta^{r+1}}{\|\mathbf{v}^{2r}\|} + (\theta_a^*(\kappa))^2\|\mathbf{g}\|_\infty, \quad (7)$$

where $\mathbf{g}$ is defined by (4) (see p.3).

Let us first consider the case $\delta_a(\mathbf{x}^*\mathbf{s}^*) > \tau$, i.e., $\max(\mathbf{x}^*\mathbf{s}^*) > \tau^2 \min(\mathbf{x}^*\mathbf{s}^*)$. From inequalities (6) and (7) one has

$$\tau^2 \left( \alpha - \theta_a^*(\kappa) \frac{\alpha^{r+1}}{\|\mathbf{v}^{2r}\|} - (\theta_a^*(\kappa))^2 \|\mathbf{g}\|_\infty \right) < \beta - \theta_a^*(\kappa) \frac{\beta^{r+1}}{\|\mathbf{v}^{2r}\|} + (\theta_a^*(\kappa))^2 \|\mathbf{g}\|_\infty.$$

Recalling the equality $\alpha\tau^2 = \beta$, and dividing both sides of the inequality by $\theta_a^*(\kappa)$, one gets

$$\frac{\beta^r - \alpha^r}{\|\mathbf{v}^{2r}\|} < \theta_a^*(\kappa) \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \|\mathbf{g}\|_\infty. \qquad (8)$$

If $\theta_a^*(\kappa)$ is substituted by $\theta_3 = \frac{4(\tau^{2r}-1)}{(1+4\kappa)(1+\tau^2)\tau^{2r}\sqrt{n}}$, the right hand side of inequality (8) increases, so the inequality is still true. After substitution one has

$$\frac{1+4\kappa}{4} \beta < \|\mathbf{g}\|_\infty. \qquad (9)$$

Since $\mathbf{v}^2 \le \beta\mathbf{e}$ and $\mathbf{g} = \Delta\mathbf{x}\Delta\mathbf{s}$ (see the notation given by (4)), inequality (9) gives

$$\|\mathbf{v}\|_\infty^2 \le \beta < \frac{4}{1+4\kappa} \|\mathbf{g}\|_\infty = \frac{4}{1+4\kappa} \|\Delta\mathbf{x}\Delta\mathbf{s}\|_\infty. \qquad (10)$$

One can check, that

$$\|\mathbf{v}^{2r+1}\|^2 \le \|\mathbf{v}\|_\infty^2 \|\mathbf{v}^{2r}\|^2. \qquad (11)$$

Since $(\Delta\mathbf{x}, \Delta\mathbf{s})$ is the solution of system (3) with $\mathbf{a} = -\mathbf{v}^{2r+2}/\|\mathbf{v}^{2r}\|$, by inequalities (10) and (11) we have

$$\left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}} \right\|^2 = \left\| \frac{\mathbf{v}^{2r+1}}{\|\mathbf{v}^{2r}\|} \right\|^2 \le \|\mathbf{v}\|_\infty^2.$$

Therefore, by the second statement of Lemma 11, we get that inequality (10) contradicts to the $\mathcal{P}_*(\kappa)$ property and vector $\Delta\mathbf{x}$ is a certificate for this fact.

Now we consider the case $(\mathbf{x}^*, \mathbf{s}^*)$ is not feasible, so there exists such an index $i$, that either $x_i^* < 0$ or $s_i^* < 0$. Let us consider the maximum feasible step size $\hat{\theta} < \theta_a^*(\kappa)$, for which $(\mathbf{x}(\hat{\theta}), \mathbf{s}(\hat{\theta})) = (\hat{\mathbf{x}}, \hat{\mathbf{s}}) \ge \mathbf{0}$ holds and at least one of its coordinates is 0. For this point $\hat{\mathbf{x}}\,\hat{\mathbf{s}} \ne \mathbf{0}$, else $\bar{\theta} = \hat{\theta}$ by the definition of $\bar{\theta}$, and the new point would be an exact solution, so the decrease of the complementarity gap would be $\mathbf{x}^T\mathbf{s}$ contradicting with the assumption of the lemma. Therefore $0 \ne \max(\hat{\mathbf{x}}\,\hat{\mathbf{s}}) > \tau^2 \min(\hat{\mathbf{x}}\,\hat{\mathbf{s}}) = 0$, so inequality (8) holds with $\hat{\theta}$. Because of $\theta_3 \ge \theta_a^*(\kappa) > \hat{\theta}$, inequality (9) holds as well, and as we have already seen this means that the matrix $M$ is not $\mathcal{P}_*(\kappa)$ and the vector $\Delta\mathbf{x}$ is a certificate for this fact. ∎

The following lemma proves, that the algorithm is well defined.

**Lemma 13** *At each iteration, when the value of $\kappa$ is updated, then the new value of $\theta_a^*(\kappa)$ satisfies the inequality $\mathbf{x}(\bar{\theta})^T\mathbf{s}(\bar{\theta}) \le (1 - 0.25\,\nu\,\theta_a^*(\kappa))\mathbf{x}^T\mathbf{s}$.*

**Proof:** In the proof of Theorem 6.1 in [11] (see Theorem 21 in the Appendix) the $\mathcal{P}_*(\kappa)$ property is only used for the vector $\Delta\mathbf{x}$. When parameter $\kappa$ is updated, then we choose the new value in such a way, that the inequality in the definition of $\mathcal{P}_*(\kappa)$-matrices (2) would hold for vector $\Delta\mathbf{x}$. Therefore the new point defined by the updated value of step size $\theta_a^*(\kappa)$ is feasible and it is in the $\tau$-neighborhood of the central path. Thus the new value of $\theta_a^*(\kappa)$ was considered in the definition of $\bar{\theta}$, so $\mathbf{x}(\bar{\theta})^T\mathbf{s}(\bar{\theta}) \le (1 - 0.25\,\nu\,\theta_a^*(\kappa))\,\mathbf{x}^T\mathbf{s}$. ∎

Now we are ready to state the complexity result for the modified affine scaling algorithm for general LCPs in case an initial interior point is given.

**Theorem 14** *Let $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{F}^0$ such that $\delta_a(\mathbf{x}^0\mathbf{s}^0) \le \tau = \sqrt{2}$. Then after at most $p$ iterations the affine scaling algorithm either yields a vector $(\hat{\mathbf{x}}, \hat{\mathbf{s}})$ such that $\hat{\mathbf{x}}^T\hat{\mathbf{s}} \le \varepsilon$ and $\delta_a(\hat{\mathbf{x}}\hat{\mathbf{s}}) \le \tau$, or it gives a polynomial size certificate that the matrix is not $\mathcal{P}_*(\tilde{\kappa})$, where $\hat{\kappa} \le \tilde{\kappa}$ is the largest value of parameter $\kappa$ and*

$$p = \begin{cases} \mathcal{O}\left( \frac{n(1+4\hat{\kappa})}{1-2^{-r}} \log \frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon} \right), & \textit{if } \ 0 < r \le 1, n \ge 4 \\ \mathcal{O}\left( n(1+4\hat{\kappa}) \log \frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon} \right), & \textit{if } \ r = 1, n \ge 4 \\ \mathcal{O}\left( 2^{2r-2} n(1+4\hat{\kappa}) \log \frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon} \right), & \\ \qquad \textit{if } \ 1 < r \textit{ and } n \textit{ sufficiently large.} \end{cases}$$

**Proof:** The algorithm at each iteration either takes a step, or detects, that the matrix is not $\mathcal{P}_*(\tilde{\kappa})$ and stops. If we take a Newton step, then by the definition of the algorithm and by Lemma 13 the decrease of the complementarity gap is at least $0.25\,\nu\,\theta_a^*(\kappa)\,\mathbf{x}^T\mathbf{s}$. One can see from the definition of $\theta_a^*(\kappa)$ that larger $\kappa$ means smaller $\theta_a^*(\kappa)$, so smaller lower bound on the decrease of the complementarity gap. Therefore, if the algorithm stops with an $\varepsilon$-optimal solution, then each Newton step decreases the complementarity gap by more than $0.25\,\nu\,\theta_a^*(\hat{\kappa})\,\mathbf{x}^T\mathbf{s}$. It means that after at most as many steps as in the original method the complementarity gap decreases below $\varepsilon$ in case for each vector during the algorithm sufficient decrease of the complementarity gap is realized according to the $\mathcal{P}_*(\hat{\kappa})$ property or at an earlier iteration the lack of $\mathcal{P}_*(\tilde{\kappa})$-property is detected. This observation, combined with the complexity theorem of the original algorithm, see Theorem 6.1 in [11] (see also Theorem 22 in the Appendix) proves our statement. ■

At the end of this subsection let us note that the case $r = 0$ can be treated analogously.

### 4.2. *Predictor-corrector IPMs*

In this section we modify the algorithm proposed in [18]. In this predictor-corrector algorithm we take affine and centering steps alternately. In a predictor step $\theta_p^*(\kappa)$ (see the definition in Lemma 15) is a theoretically feasible step length if the matrix $M$ is $\mathcal{P}_*(\kappa)$. Therefore, if the maximal feasible step length is smaller than $\theta_p^*(\kappa)$, then the matrix is not $\mathcal{P}_*(\kappa)$ with the actual value of $\kappa$, so $\kappa$ should be increased. In a corrector step we return to the smaller $\mathcal{D}(\gamma)$ neighborhood with step size $\theta_c^*(\kappa)$ (see the definition in Lemma 16) if the matrix is $\mathcal{P}_*(\kappa)$. Accordingly, if the new point with step length $\theta_c^*(\kappa)$ is not in $\mathcal{D}(\gamma)$, then the matrix $M$ is not $\mathcal{P}_*(\kappa)$ with the actual value of $\kappa$, so $\kappa$ should be updated. Similarly to the Affine Scaling algorithm, if in a predictor or corrector step the new value of $\kappa$ cannot be defined by (5), then the matrix is not $\mathcal{P}_*$ and the current Newton direction is a certificate of it. Furthermore, if the new value of $\kappa$ is larger than $\tilde{\kappa}$, then the matrix is not $\mathcal{P}_*(\tilde{\kappa})$ and the Newton direction is a certificate for it. The modified algorithm is summarized in Algorithm 2.

Potra and Liu determined the maximum feasible predictor step length as the minimum of $n+1$ number [18] ($\bar{\theta} = \min\{\bar{\theta}_i : 0 \le i \le n\}$ see Lemma 23 in the Appendix). Furthermore, they proved, that if the matrix $M$ is a $\mathcal{P}_*(\kappa)$-matrix, then $\theta_p^*(\kappa)$ and $\theta_c^*(\kappa)$ (defined in the following lemmas) give a feasible predictor and corrector step length pair. The following lemmas show that if either $\theta_p^*(\kappa)$ or $\theta_c^*(\kappa)$ is not a feasible step length, than the matrix is not a $\mathcal{P}_*(\kappa)$-matrix.

**Lemma 15** *If there exists an index $i$ $(0 \le i \le n)$ such that*

$$\bar{\theta}_i \le \theta_p^*(\kappa) := \frac{2\sqrt{(1-\gamma)\gamma}}{(1+4\kappa)n+2},$$

*then matrix $M$ is not a $\mathcal{P}_*(\kappa)$-matrix and the affine Newton direction is a certificate for this.*

**Proof:** For any $\kappa \ge 0$ and $n \ge 1$

$$\theta_p^*(\kappa) < \frac{2}{1+\sqrt{1+4\kappa}},$$

therefore if $\bar{\theta}_0 \le \theta_p^*(\kappa)$, then by the definition of $\bar{\theta}_0$ one has

$$\frac{2}{1+\sqrt{1-4\mathbf{e}^T\mathbf{g}/n}} = \bar{\theta}_0 < \frac{2}{1+\sqrt{1+4\kappa}},$$

implying $\mathbf{e}^T\mathbf{g}/n < -\kappa$, thus $\sum_{i\in I}\Delta x_i \Delta s_i < -\kappa n\mu = -\kappa\, \mathbf{x}^T\mathbf{s}$. Therefore, by Lemma 11 the matrix $M$ is not a $\mathcal{P}_*(\kappa)$-matrix and the affine Newton direction $\Delta\mathbf{x}$ is a certificate for this.

If $\bar{\theta}_i \le \theta_p^*(\kappa)$, where $0 < i \le n$, then let consider the following inequality, which was proved by Potra and Liu in [18] on p.158:

$$\sqrt{(1-\gamma)\gamma} + \sqrt{((1+4\kappa)n+1)^2 + \gamma(1-\gamma)} < (1+4\kappa)n+2. \tag{12}$$

Based on the proof of [17, Thm 3.3] (see Lemma 24 in the Appendix), Lemma 8 and the definition of $t$, one has

$$\begin{aligned}\frac{2\sqrt{(1-\gamma)\gamma}}{(1+4\kappa)n+2} &= \theta_p^*(\kappa) \ge \bar{\theta}_i \\ &\ge \frac{2}{1+\sqrt{1+(t\gamma)^{-1}(4\|\mathbf{g}\|_\infty + 4\mathbf{e}^T\mathbf{g}/n)}} \\ &\ge \frac{2}{1+\sqrt{1+(t\gamma)^{-1}(4\|\mathbf{g}\|_\infty+1)}} \\ &= \frac{2\sqrt{(1-\gamma)\gamma}}{\sqrt{(1-\gamma)\gamma}+\sqrt{((1+4\kappa)n+1)(4\|\mathbf{g}\|_\infty+1)+\gamma(1-\gamma)}}.\end{aligned} \tag{13}$$

From inequality (13) and (12) we get

$$4\|\mathbf{g}\|_\infty + 1 > (1+4\kappa)n+1. \tag{14}$$

Since $(\Delta\mathbf{x}, \Delta\mathbf{s})$ is a solution of system (3) with $\mathbf{a} = -\mathbf{xs}$, and using inequality (14) with $\mu\, n = \mathbf{x}^T\mathbf{s}$, one has

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_\infty > \frac{(1+4\kappa)}{4}\,\mathbf{x}^T\mathbf{s} = \frac{1+4\kappa}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2,$$

so by the second statement of Lemma 11 one has $M \notin \mathcal{P}_*(\kappa)$, and $\Delta\mathbf{x}$ is a certificate for this. ■

Now let us analyze the corrector step.

**Lemma 16** *If $\theta_c^*(\kappa) := \frac{2\gamma}{(1+4\kappa)n+1}$ is such a corrector step length that $(\bar{\mathbf{x}}(\theta_c^*(\kappa)), \bar{\mathbf{s}}(\theta_c^*(\kappa))) \notin \mathcal{D}(\gamma)$, then the matrix $M$ is not a $\mathcal{P}_*(\kappa)$-matrix and the corrector Newton direction is a certificate for this.*

**Proof:** Notice that

$$\bar{\mathbf{x}}(\theta)\bar{\mathbf{s}}(\theta) = (1-\theta)\bar{\mathbf{x}}\bar{\mathbf{s}} + \theta\bar{\mu}\,\mathbf{e} + \theta^2\Delta\bar{\mathbf{x}}\Delta\bar{\mathbf{s}}$$

**Algorithm 2 - Predictor-corrector algorithm**

**Input:**
an upper bound $\tilde{\kappa} > 0$ on the value of $\kappa$;
an accuracy parameter $\varepsilon > 0$;
a proximity parameter $\gamma \in (0, 1)$;
an initial point $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{D}(\gamma)$;
**begin**
$\mathbf{x} := \mathbf{x}^0$, $\mathbf{s} := \mathbf{s}^0$, $\mu := (\mathbf{x}^0)^T \mathbf{s}^0 / n$, $\kappa := 0$;
**while** $\mathbf{x}^T \mathbf{s} \geq \varepsilon$ **do**
*Predictor step*
$t = \frac{1-\gamma}{(1+4\kappa)n+1}$;
calculate the affine Newton direction $(\Delta\mathbf{x}, \Delta\mathbf{s})$ with $\mathbf{a} = -\mathbf{x}\mathbf{s}$;
**if** (the Newton direction does not exists, or it is not unique) **then**
**return** the matrix is not $\mathcal{P}_0$; *% see Corollary 6*
$\bar{\theta} = \sup \left\{ \hat{\theta} > 0 : \ (\mathbf{x}(\theta), \mathbf{s}(\theta)) \in \mathcal{D}\big((1-\mathrm{t})\gamma\big), \ \forall\, \theta \in [0, \hat{\theta}] \right\}$;
**if** $\big(\bar{\theta} < \theta_p^*(\kappa)\big)$ **then**
calculate $\kappa(\Delta\mathbf{x})$; *% see (5)*
**if** ($\kappa(\Delta\mathbf{x})$ is not defined) **then**
**return** the matrix is not $\mathcal{P}_*$; *% see Lemma 10*
**if** ($\kappa(\Delta\mathbf{x}) > \tilde{\kappa}$) **then**
**return** the matrix is not $\mathcal{P}_*(\tilde{\kappa})$; *% see Lemma 9*
$\kappa = \kappa(\Delta\mathbf{x})$;
update $\theta_p^*(\kappa)$ and $\theta_c^*(\kappa)$;
$\bar{\mathbf{x}} = \mathbf{x}(\bar{\theta}), \ \bar{\mathbf{s}} = \mathbf{s}(\bar{\theta}), \ \bar{\mu} = \bar{\mathbf{x}}^T \bar{\mathbf{s}} / n$;
*Corrector step*
calculate the centering Newton direction $(\Delta\bar{\mathbf{x}}, \Delta\bar{\mathbf{s}})$ with $\mathbf{a} = \mu\mathbf{e} - \bar{\mathbf{x}}\bar{\mathbf{s}}$;
**if** (the Newton direction does not exists, or it is not unique) **then**
**return** the matrix is not $\mathcal{P}_0$; *% see Corollary 6*
**if** $\big( (\bar{\mathbf{x}}(\theta_c^*(\kappa)), \bar{\mathbf{x}}(\theta_c^*(\kappa))) \notin \mathcal{D}(\gamma) \big)$
calculate $\kappa(\Delta\bar{\mathbf{x}})$; *% see (5)*
**if** ($\kappa(\Delta\bar{\mathbf{x}})$ is not defined) **then**
**return** the matrix is not $\mathcal{P}_*$; *% see Lemma 10*
**if** ($\kappa(\Delta\mathbf{x}) > \tilde{\kappa}$) **then**
**return** the matrix is not $\mathcal{P}_*(\tilde{\kappa})$; *% see Lemma 9*
$\kappa = \kappa(\Delta\bar{\mathbf{x}})$;
update $\theta_p^*(\kappa)$ and $\theta_c^*(\kappa)$;
$\theta^+ = \operatorname{argmin} \{ \bar{\mu}(\theta) : \ (\bar{\mathbf{x}}(\theta), \bar{\mathbf{s}}(\theta)) \in \mathcal{D}(\gamma) \}$;
$\mathbf{x}^+ = \bar{\mathbf{x}} + \theta^+ \Delta\bar{\mathbf{x}}, \ \mathbf{s}^+ = \bar{\mathbf{s}} + \theta^+ \Delta\bar{\mathbf{s}}, \ \mu^+ = (\mathbf{x}^+)^T \mathbf{s}^+ / n$;
$\mathbf{x} = \mathbf{x}^+, \ \mathbf{s} = \mathbf{s}^+, \ \mu = \mu^+$;
**end**
**end.**

and

$$\bar\mu(\theta) = \bar\mu + \theta^2 \frac{\Delta\bar{\mathbf{x}}^T \Delta\bar{\mathbf{s}}}{n}.$$

From Lemma 7 and the proof of [17, Thm 3.3] (see Lemma 25 in the Appendix) we get

$$\begin{aligned}\Delta\bar{\mathbf{x}}^T\Delta\bar{\mathbf{s}} &\le \sum_{\mathcal{I}_+} \Delta\bar x_i \Delta\bar s_i \le \frac{1}{4}\left\|\frac{\bar\mu\mathbf{e} - \bar{\mathbf{x}}\bar{\mathbf{s}}}{\sqrt{\bar{\mathbf{x}}\bar{\mathbf{s}}}}\right\|^2 \\ &= \frac{1}{4}\,\bar\mu \left\|\sqrt{\frac{\bar{\mathbf{x}}\bar{\mathbf{s}}}{\bar\mu}} - \sqrt{\frac{\bar\mu}{\bar{\mathbf{x}}\bar{\mathbf{s}}}}\right\|^2 \\ &\le \frac{1}{4}\,\bar\mu\,\frac{1-(1-t)\gamma}{(1-t)\gamma}\,n,\end{aligned}$$

therefore

$$\bar\mu(\theta) \le \left(1 + \frac{1-(1-t)\gamma}{4(1-t)\gamma}\,\theta^2\right)\bar\mu. \tag{15}$$

Since $\theta_c^*(\kappa)$ is an infeasible step length, there exists an index $i$ such that

$$\bar x(\theta_c^*(\kappa))_i\ \bar s(\theta_c^*(\kappa))_i < \gamma\bar\mu(\theta_c^*(\kappa)), \quad \text{namely}$$

$$(1-\theta_c^*(\kappa))\bar x_i \bar s_i + \theta_c^*(\kappa)\bar\mu \ + (\theta_c^*(\kappa))^2 \Delta\bar x_i \Delta\bar s_i < \gamma\bar\mu(\theta_c^*(\kappa)).$$

The predictor point $(\bar{\mathbf{x}}, \bar{\mathbf{s}}) \in \mathcal{D}((1-t)\gamma)$, so $\bar x_i \bar s_i \ge (1-t)\gamma\bar\mu$. Furthermore, by inequality (15) one has

$$(1-\theta_c^*(\kappa))(1-t)\gamma\bar\mu + \theta_c^*(\kappa)\bar\mu + (\theta_c^*(\kappa))^2\Delta\bar x_i\Delta\bar s_i < \gamma\left(1 + \frac{1-(1-t)\gamma}{4(1-t)\gamma}\,(\theta_c^*(\kappa))^2\right)\bar\mu,$$

which implies

$$(\theta_c^*(\kappa))^2 \frac{\Delta\bar x_i \Delta\bar s_i}{\bar\mu} < t\gamma - \theta_c^*(\kappa)(1-(1-t)\gamma) + \frac{1-(1-t)\gamma}{4(1-t)}(\theta_c^*(\kappa))^2. \tag{16}$$

One can check, the following equality by substituting the values of $t$ and $\theta_c^*(\kappa)$

$$0 \le \frac{(1-\gamma)\gamma^2}{((1+4\kappa)n+1)^2} = -t\gamma + \theta_c^*(\kappa)(1-(1-t)\gamma) - \frac{1-(1-t)\gamma}{4(1-t)\gamma}\big[(1+4\kappa)n + \gamma\big](\theta_c^*(\kappa))^2.$$

Therefore

$$-\frac{1-(1-t)\gamma}{4(1-t)\gamma}\,(1+4\kappa)\,n\,(\theta_c^*(\kappa))^2 \ge t\gamma - \theta_c^*(\kappa)(1-(1-t)\gamma) + \frac{1-(1-t)\gamma}{4(1-t)}\,(\theta_c^*(\kappa))^2.$$

Combining this with inequality (16), and then considering the proximity measure estimation in the proof of [17, Thm 3.3] (see Lemma 25 in the Appendix), we get

$$\begin{aligned}\Delta\bar x_i\Delta\bar s_i &< -\frac{1-(1-t)\gamma}{4(1-t)\gamma}\,(1+4\kappa)\,n\bar\mu \\ &\le -\frac{(1+4\kappa)\bar\mu}{4}\left\|\sqrt{\frac{\bar{\mathbf{x}}\bar{\mathbf{s}}}{\bar\mu}} - \sqrt{\frac{\bar\mu}{\bar{\mathbf{x}}\bar{\mathbf{s}}}}\right\|^2.\end{aligned} \tag{17}$$

Since $(\Delta\bar{\mathbf{x}}, \Delta\bar{\mathbf{s}})$ is a solution of system (3) with $\mathbf{a} = \bar\mu\mathbf{e} - \bar{\mathbf{x}}\bar{\mathbf{s}}$, using inequality (17), one get

$$\begin{aligned}\|\Delta\bar{\mathbf{x}}\Delta\bar{\mathbf{s}}\|_\infty &> \frac{(1+4\kappa)\bar\mu}{4}\left\|\sqrt{\frac{\bar{\mathbf{x}}\bar{\mathbf{s}}}{\bar\mu}} - \sqrt{\frac{\bar\mu}{\bar{\mathbf{x}}\bar{\mathbf{s}}}}\right\|^2 \\ &= \frac{1+4\kappa}{4}\left\|\frac{\mathbf{a}}{\sqrt{\bar{\mathbf{x}}\bar{\mathbf{s}}}}\right\|^2.\end{aligned}$$

Thus, by the second statement of Lemma 11, the matrix $M$ is not a $\mathcal{P}_*(\kappa)$-matrix and the corrector Newton direction $\Delta\bar{\mathbf{x}}$ is a certificate for this. ■

The following lemma proves, that the predictor-corrector algorithm is well defined.

**Lemma 17** *At each iteration when the value of $\kappa$ is updated, the new value of $\theta_p^*(\kappa)$ satisfies the inequality $\bar\theta \ge \theta_p^*(\kappa)$, and the new point $(\bar{\mathbf{x}}(\theta_c^*(\kappa)), \bar{\mathbf{s}}(\theta_c^*(\kappa)))$, determined by the new value of the corrector step size $\theta_c^*(\kappa)$, is in the $\mathcal{D}(\gamma)$ neighborhood.*

**Proof:** In the proof of [17, Thm 3.3] (see Lemma 26 in the Appendix) we use the $\mathcal{P}_*(\kappa)$ property only for the vector $\Delta\mathbf{x}$ or $\Delta\bar{\mathbf{x}}$. When parameter $\kappa$ is updated, then we choose the new value in such a way that the inequality in the definition of $\mathcal{P}_*(\kappa)$-matrices (2) holds for the vectors $\Delta\mathbf{x}$ and $\Delta\bar{\mathbf{x}}$. Therefore the new value of $\theta_p^*(\kappa)$ satisfies the inequality $\bar\theta \ge \theta_p^*(\kappa)$, and the new value of $\theta_c^*(\kappa)$ determines a point in the $\mathcal{D}(\gamma)$ neighborhood. ■

Now we are ready to state the complexity result for the modified predictor-corrector algorithm for general LCPs in case an initial interior point is available.

**Theorem 18** *Let $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{F}^0$ such that $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{D}(\gamma)$. Then after at most*

$$\mathcal{O}\left((1+\hat\kappa)n\log\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon}\right)$$

*steps, where $\hat{\kappa} \leq \tilde{\kappa}$ is the largest value of parameter $\kappa$ throughout the algorithm, the predictor-corrector algorithm generate a point $(\hat{\mathbf{x}}, \hat{\mathbf{s}})$, such that $\hat{\mathbf{x}}^T\hat{\mathbf{s}} \leq \varepsilon$ and $(\hat{\mathbf{x}}, \hat{\mathbf{s}}) \in \mathcal{D}(\gamma)$, or provides a certificate that the matrix is not $\mathcal{P}_*(\tilde{\kappa})$.*

**Proof:** We follow the proof of the previous complexity theorem (see Theorem 14). If we take a predictor and a corrector step, then by Theorem 27 and Lemma 17 the decrease of the complementarity gap is at least

$$\frac{3\sqrt{(1-\gamma)\gamma}}{2((1+4\kappa)n+2)}\frac{\mathbf{x}^T\mathbf{s}}{n}.$$

This expression is a decreasing function of $\kappa$, so at each iteration, when we make a predictor and a corrector step, the complementarity gap decreases at least by

$$\frac{3\sqrt{(1-\gamma)\gamma}}{2((1+4\tilde{\kappa})n+2)}\frac{\mathbf{x}^T\mathbf{s}}{n}.$$

We take at most as many iterations as in the original predictor-corrector IPM with a $\mathcal{P}_*(\hat{\kappa})$-matrix. Thus, referring to the complexity theorem of the original algorithm (see Theorem 28 in the Appendix) we have proved the theorem. ■

## 5. Summary

In this paper we have presented modifications of affine scaling and the predictor-corrector interior point algorithms that enable us to solve general LCPs without the prerequisite to verify special properties of the coefficient matrix. In particular, we have given two constructive proofs of the following EP type theorem from paper [9]. We assume that the data are rational (solving problems with computer this is a reasonable assumption), ensuring polynomial encoding size of certificates and polynomial complexity of the algorithms.

**Theorem 19** *Let an arbitrary matrix $M \in \mathbb{Q}^{n\times n}$, a vector $\mathbf{q} \in \mathbb{Q}^n$ and a point $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{F}^0$ be given. Then one can verify in polynomial time that at least one of the following statements hold*

*(1) problem LCP has a feasible complementary solution $(\mathbf{x}, \mathbf{s})$ whose encoding size is polynomially bounded.*

*(2) the matrix $M$ is not in the class of $\mathcal{P}_*(\tilde{\kappa})$ and there is a certificate whose encoding size is polynomially bounded.*

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## 6. Appendix

To make our paper self contained, we include those results from [11,18] that are needed for our developments. All lemmas, theorems are converted to our notations.

**Lemma 20** *(Lemma 4.3 in [12]) Let $M$ be an arbitrary real matrix, $\delta_a(\mathbf{xs}) < \tau$ and let $(\Delta\mathbf{x}, \Delta\mathbf{s})$ be the affine scaling direction.*

*(i) If $0 \le r \le 1$ and $\theta \le \frac{2}{\sqrt{n}}$, then*

$$\mathbf{x}(\theta)^T\mathbf{s}(\theta) \le \left(1 - \frac{\theta}{2\sqrt{n}}\right)\|\mathbf{v}\|^2.$$

*(ii) If $1 \le r$ and $\theta \le \frac{2\tau^{2-2r}}{\sqrt{n}}$, then*

$$\mathbf{x}(\theta)^T\mathbf{s}(\theta) \le \left(1 - \frac{\theta\tau^{2-2r}}{2\sqrt{n}}\right)\|\mathbf{v}\|^2.$$

**Theorem 21** *(Theorem 6.1 in [11]) Let $M$ be a $\mathcal{P}_*(\kappa)$-matrix, $r > 0$, $\tau > 1$ and let $(\Delta\mathbf{x}, \Delta\mathbf{s})$ be the affine scaling direction. If $(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^0$, $\delta_a(\mathbf{xs}) \le \tau$, $0 \le \theta$ and*

$$\theta \le \min\left\{\frac{2}{(1+4\kappa)\tau}\left(\sqrt{1+4\kappa+\frac{1}{\tau^2 n}} - \frac{1}{\tau\sqrt{n}}\right),\right.$$
$$\left.\frac{\sqrt{n}}{(r+1)\tau^{2r}},\ \frac{4(\tau^{2r}-1)}{(1+4\kappa)(1+\tau^2)\tau^{2r}\sqrt{n}}\right\},$$

*then $(\mathbf{x}(\theta), \mathbf{s}(\theta))$ is strictly feasible and $\delta_a(\mathbf{x}(\theta)\mathbf{s}(\theta)) \le \tau$.*

**Theorem 22** *(Corollary 6.1 in [11])*
*Let $M \in \mathcal{P}_*(\kappa)$ and $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{F}^0$ such that $\delta_a(\mathbf{x}^0\mathbf{s}^0) \le \tau = \sqrt{2}$.*

- *If $0 < r \le 1$ and $n \ge 4$, then we may choose $\theta = \frac{4(1-2^{-r})}{3(1+4\kappa)\sqrt{n}}$, hence the complexity of the affine scaling algorithm is $\mathcal{O}\left(\frac{n(1+4\kappa)}{1-2^{-r}}\log\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon}\right)$.*
- *If $r = 1$ and $n \ge 4$, then we may choose $\theta = \frac{1}{2(1+4\kappa)\sqrt{n}}$, hence the complexity of the affine scaling algorithm is $\mathcal{O}\left(n(1+4\kappa)\log\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon}\right)$.*
- *If $r > 1$ and $n$ is sufficiently large, then we may choose $\theta = \frac{1}{2^r(1+4\kappa)\sqrt{n}}$, hence the complexity of the affine scaling algorithm is $\mathcal{O}\left(2^{2r-2}n(1+4\kappa)\log\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon}\right)$.*

**Lemma 23** *(From expressions (3.16), (3.17) in [18])*
*Let $M$ be an arbitrary matrix, $(\mathbf{x}, \mathbf{s}) \in \mathcal{D}(\gamma)$, $(\Delta\mathbf{x}, \Delta\mathbf{s})$ be the predictor direction in the predictor-corrector algorithm and let the predictor step length be $\bar{\theta}$*

$$= \sup\left\{\hat{\theta} > 0 : (\mathbf{x}(\theta), \mathbf{s}(\theta)) \in \mathcal{D}\big((1-\mathrm{t})\gamma\big), \forall\, \theta \in [0, \hat{\theta}]\right\}.$$

*Furthermore, let* $\bar{\theta}_0 = \dfrac{2}{1+\sqrt{1-4\mathbf{e}^T\mathbf{g}/n}}$ *and*

$$\bar{\theta}_i = \begin{cases} \infty & \text{if } \Delta_i \le 0 \\ 1 & \text{if } g_i - (1-t)\gamma\mathbf{e}^T\mathbf{g}/n = 0 \\ \frac{2(v_i^2-(1-t)\gamma)}{v_i^2-(1-t)\gamma+\sqrt{\Delta_i}} & \text{if } \Delta_i > 0 \text{ and} \\ & g_i - (1-t)\gamma\mathbf{e}^T\mathbf{g}/n \ne 0, \end{cases}$$

*where*

$$\Delta_i = (v_i^2 - (1-t)\gamma)^2 - 4(v_i^2 - (1-t)\gamma)\,(g_i - (1-t)\gamma\mathbf{e}^T\mathbf{g}/n)\,, \quad \text{for each } 0 < i \le n.$$

*Then we have*

$$\bar{\theta} = \min\left\{\bar{\theta}_i : 0 \le i \le n\right\}.$$

**Lemma 24** *(From the proof of Theorem 3.3 in [18])* *Let the assumptions of Lemma 23 hold, and $\bar{\theta}_i$, $1 \le i \le n$ be as it is given in Lemma 23, then*

$$\bar{\theta}_i \ge \frac{2}{1+\sqrt{1+(t\gamma)^{-1}(4\|\mathbf{g}\|_\infty + 4\mathbf{e}^T\mathbf{g}/n)}}.$$

**Lemma 25** *(From the proof of Theorem 3.3 in [18])* *Let $M$ be an arbitrary matrix and let the point after the predictor step in the predictor-corrector algorithm satisfy $(\bar{\mathbf{x}}, \bar{\mathbf{s}}) \in \mathcal{D}((1-t)\gamma)$. Then*

$$\left\|\sqrt{\frac{\bar{\mathbf{x}}\bar{\mathbf{s}}}{\bar{\mu}}} - \sqrt{\frac{\bar{\mu}}{\bar{\mathbf{x}}\bar{\mathbf{s}}}}\right\|^2 \le \frac{1-(1-t)\gamma}{(1-t)\gamma}\, n.$$

**Lemma 26** *(From Theorem 3.3 in [18])* *Let $M$ be a $\mathcal{P}_*(\kappa)$-matrix and $(\mathbf{x}, \mathbf{s}) \in \mathcal{D}(\gamma)$. Then the predictor step length satisfy*

$$\theta_p^*(\kappa) := \frac{2\sqrt{(1-\gamma)\gamma}}{(1+4\kappa)n+2} \le \sup\Big\{\hat{\theta} > 0 : (\mathbf{x}(\theta), \mathbf{s}(\theta)) \in \mathcal{D}\big((1-t)\gamma\big), \forall\, \theta \in [0, \hat{\theta}]\Big\},$$

*and the corrector step length*

$$\theta_c^*(\kappa) := \frac{2\gamma}{(1+4\kappa)n+1}$$

*determines a point in the $\mathcal{D}(\gamma)$ neighborhood, i.e., $(\bar{\mathbf{x}}(\theta_c^*(\kappa)), \bar{\mathbf{s}}(\theta_c^*(\kappa))) \in \mathcal{D}(\gamma)$, where $(\bar{\mathbf{x}}, \bar{\mathbf{s}}) = (\mathbf{x}(\theta_p^*(\kappa)), \mathbf{s}(\theta_p^*(\kappa))) \in \mathcal{D}((1-t)\gamma)$.*

**Lemma 27** *(From Theorem 3.3 in [18])* *Let $M$ be an arbitrary matrix, $(\mathbf{x}, \mathbf{s}) \in \mathcal{D}(\gamma)$, $\mu_g = \mathbf{x}^T\mathbf{s}/n$, the definition of parameters $\theta_p^*(\kappa)$ and $\theta_c^*(\kappa)$ be the same as in Lemma 26, $\bar{\theta}$ be the predictor and $\theta^+$ be the corrector step length, $(\Delta\mathbf{x}, \Delta\mathbf{s})$ be the predictor and $(\Delta\bar{\mathbf{x}}, \Delta\bar{\mathbf{s}})$ the corrector Newton direction in the predictor-corrector algorithm. If $\bar{\theta} \ge \theta_p^*(\kappa)$ and the step length $\theta_c^*(\kappa)$ determines a point in the $\mathcal{D}(\gamma)$ neighborhood, i.e., $(\bar{\mathbf{x}}(\theta_c^*(\kappa)), \bar{\mathbf{s}}(\theta_c^*(\kappa))) \in \mathcal{D}(\gamma)$, where $(\bar{\mathbf{x}}, \bar{\mathbf{s}}) = (\mathbf{x}(\bar{\theta}), \mathbf{s}(\bar{\theta}))$, then*

$$\mu_g^+ \le \left(1 - \frac{3\sqrt{(1-\gamma)\gamma}}{2((1+4\kappa)n+2)}\right)\mu_g,$$

*where $\mu_g^+ = (\mathbf{x}^+)^T\mathbf{s}^+/n = \bar{\mathbf{x}}(\theta^+)^T\bar{\mathbf{s}}(\theta^+)/n$.*

**Theorem 28** *(Corollary 3.4 in [18])* *Let $M$ be a $\mathcal{P}_*(\kappa)$-matrix and $(\mathbf{x}^0, \mathbf{s}^0)$ be a feasible interior point such that $(\mathbf{x}^0, \mathbf{s}^0) \in \mathcal{D}(\gamma)$. Then in at most $\mathcal{O}\left((1+\kappa)n\log\frac{(\mathbf{x}^0)^T\mathbf{s}^0}{\varepsilon}\right)$ steps the predictor-corrector algorithm produces a point $(\hat{\mathbf{x}}, \hat{\mathbf{s}})$ such that $(\hat{\mathbf{x}}, \hat{\mathbf{s}}) \in \mathcal{D}(\gamma)$ and $\hat{\mathbf{x}}^T\hat{\mathbf{s}} \le \varepsilon$.*

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